Quest for Mathematics I (E2): Exercise sheet 3 Solutions

1. Writing $f(x) = -7x^3 + 20x^2 - x + 1$, we have $f(-1) = 29$, $f(0) = 1$, $f(1) = 13$, $f(2) = 23$, $f(3) = -11$. Hence, since f is continuous, the intermediate value theorem tells us that the function has a root in the interval $[2, 3]$.

2. Note there are three possibilities: $f(0) = 0$, $f(0) > 0$ and $f(0) < 0$. In the first case, $T(0) = T(\pi)$, and so this confirms two opposite points on the equator have the same temperature. If $f(0) > 0$, then $f(\pi) = T(2\pi) - T(\pi) = -(T(\pi) - T(0)) = -f(0) < 0$. Hence, by the intermediate value theorem, there exists a $\theta \in (0, \pi)$ such that $f(\theta) = 0$. For this choice of θ, we have $T(\theta + \pi) = T(\theta)$, which is the desired result. For $f(0) < 0$, we can argue similarly to the previous case.

3. (a) For $x \neq 0$, we have
$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2}{x|x|} = \frac{x}{|x|}.$$
Hence
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1,$$
and
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1,$$
so f is not differentiable at 0.

   (b) For $x \neq 0$, we have
$$\frac{f(x) - f(0)}{x - 0} = \frac{x|x|^2}{x|x|} = |x|.$$
Hence
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0,$$
so $f'(0) = 0$.

   (c) We have that
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} x = 0,$$
and
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} 1 = 1,$$
so f is not differentiable at 0.

   (d) We have that
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1,$$
and
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} 1 = 1,$$
so $f'(0) = 1$.

4. (a) $f'(x) = 2x\cos(x^2)$
   (b) $f'(x) = 2\sin x \cos x = \sin(2x)$
   (c) $f'(x) = \frac{-7x^4 + 2x}{(7x^3+1)^2}$
   (d) $f'(x) = \frac{20x^2 + 16x + 1}{2\sqrt{x+1}}$
   (e) $f'(x) = \frac{x^2}{(x^3+1)^{2/3}}$